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# The Numerical Computation of Steady Vortex Pairs

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**Abstract**—We present several algorithms for the computation of the solution to some semilinear elliptic problems with discontinuous nonlinearities. These are related to the equilibrium of steady vortex pairs. We illustrate with numerical experiments. First, we consider a problem which possesses an equivalent variational formulation. Then, by analogy, we propose an algorithm in the context of a nonvariational problem.

**Keywords**—Semilinear elliptic problems, Discontinuous nonlinearities, Steady vortex pairs.

## 1. INTRODUCTION

In this paper, we are concerned with the numerical solution of some problems related to the equilibrium of steady vortex pairs in an ideal fluid. To this purpose, we will use some iterative algorithms which are very close to those in [1] (see also [2]), and rely on exact regularization. Our problems are written as semilinear elliptic systems, for which the main difficulty is the presence of a discontinuous nonlinearity.

For the solution of the first problem considered in this paper (see (2) below), the underlying idea is to introduce a variational reformulation. This reduces the task to the search of the critical points of a functional; that is, the difference of two convex functions. Then, using exact regularization, the original problems are replaced by other equivalent regular problems for which fixed point iterates are well defined.

In the case of the second problem (see (15)), unfortunately, there is no known variational equivalent formulation. However, we also present an iterative algorithm, where the specific iterates are defined by analogy with the variational case.

For the numerical solution, we use finite element approximation techniques. For the computations, as well as for the visualization of the numerical results, we have used the MODULEF finite element code (see [3]).

The plan is as follows. In Section 2, we recall a general problem modelling the equilibrium of vortex pairs in an ideal fluid. Section 3 deals with the variational case; we present several algorithms, we justify some theoretical aspects, and also, we present numerical results. Finally, in Section 4, we consider the nonvariational case.

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## 2. THE GENERAL PROBLEM

The general problem we consider is the following (see, e.g., [4]):

$$
\begin{aligned}
-\Delta u(x) \in H\left(u(x) - Wx_1 - Z\right) \quad \text{a.e. in } \Omega, \qquad x = (x_1, x_2), \\
u \in H_0^1(\Omega), \qquad \int_{\partial\Omega} |\nabla u|^2 \, dx = \eta.
\end{aligned} \tag{1}
$$

Here, $\Omega$ is the following bounded open set:

$$\Omega = \left\{x \in \mathbb{R}^2;\, x = (x_1, x_2),\ 0 < x_1 < a,\ -b < x_2 < b\right\}.$$

$H$ is the maximal monotone operator (depending on a parameter $\alpha \in \mathbb{R}$), associated to Heavyside's function

$$H(s) = \begin{cases} 0 & \text{for } s < 0, \\ [0, \alpha] & \text{for } s = 0, \\ \alpha & \text{for } s > 0; \end{cases}$$

the parameter $W$ represents the constant velocity of the fluid at infinity in the direction $Ox_2$, and $\eta$ is the (prescribed) kinetic energy of the vortex motion. For the meaning of $Z$, see below.

The equations are required to be satisfied in a subset of the half-plane $\Pi = \{x = (x_1, x_2);\ x \in \mathbb{R}^2,\ x_1 > 0\}$ because the fluid motion is supposed to be symmetric about $x_1 = 0$. In the study of steady vortex pairs, parameters $a$ and $b$ are denoted to tend to $+\infty$, in order to recover the original problem in the whole half-plane.

If we denote by $\psi$ the Stokes stream function, then $u$ is $u = \psi + Wx_1 + Z$, and represents the perturbation of $\psi$ due to the vortex motion. The previous system can be viewed as a free boundary problem. The region

$$A = \{x \in \Omega;\ x = (x_1, x_2),\ \psi(x) = u(x) - Wx_1 - Z > 0\},$$

which is half the vortex pair, is *a priori* unknown (the amount of fluid flowing between the vortex boundary $\partial A$ and the axis $x_1 = 0$ is given by $Z$).

Besides the case of a vortex pair, (1) serves to model equilibrium in various related phenomena, arising in other contexts (see, e.g., [5] and the bibliography therein). In particular, a similar situation corresponding to the equilibrium of an axisymmetric vortex ring can be found in [6].

In this paper, we are interested in solving two specific problems based on the general formulation (1). They are the following:

(A) *Free vortex velocity, with vanishing flux parameter.*
Given $\eta > 0$ and $Z = 0$, find $u$ and $W > 0$ satisfying (1).

(B) *Free flux parameter.*
Given $\eta > 0$ and $W > 0$, find $u$ and $Z > 0$ satisfying (1).

Apparently, problem (B) does not possess an equivalent variational formulation; i.e., we cannot identify the solutions to problem (B) with the critical point of a functional (see [7]).

## 3. THE VARIATIONAL CASE: FREE VORTEX VELOCITY, WITH VANISHING FLUX PARAMETER

As we have pointed out in Section 1, our problem is the following:

$$
\begin{aligned}
&\text{Find } u \in H_0^1(\Omega) \quad \text{and} \quad W \in \mathbb{R} \text{ such that} \\
&-\Delta u(x) \in H\left(u(x) - Wx_1\right) \quad \text{a.e. in } \Omega, \qquad x = (x_1, x_2), \\
&\int_\Omega |\nabla u|^2 \, dx = \eta.
\end{aligned} \tag{2}
$$

As noticed in [1], it would be interesting to rewrite (2) as a problem of the kind

$$\begin{aligned} &\min\ J^*(q) = g^*(q) - f^*(B^*q) \\ &\text{subject to } q \in \partial K, \end{aligned} \tag{3}$$

for adequate $f$, $g$, $B$, and $K$. Here, $g^*$ and $f^*$ are the conjugate of the convex, proper and l.s.c. functions $g$ and $f$, $B^*$ is the adjoint of the bounded linear operator $B$, and $\partial K$ is the boundary of the closed convex set $K$.

Thus, we take[1]

$$H = L^2(\Omega), \qquad V = H_0^1(\Omega), \qquad B: \text{the compact embedding } H_0^1(\Omega) \hookrightarrow L^2(\Omega),$$

$$K = \left\{ q \in L^2(\Omega);\ \int_\Omega (BSB^*q)\cdot q\,dx \le \eta \right\},$$

$$f: H_0^1(\Omega) \longrightarrow \mathbb{R}, \qquad f(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \quad \forall v \in H_0^1(\Omega),$$

$$g: L^2(\Omega) \longrightarrow \mathbb{R}, \qquad g(q) = \int_\Omega G(x,q)\,dx \quad \forall q \in L^2(\Omega),$$

where $S$ stands for the inverse of $-\Delta$ with homogeneous Dirichlet conditions and $G(x,s) = \alpha(s-x_1)_+ \forall (x,s) \in \Omega\times\mathbb{R}$ (so that $\partial_s G(x,s) = H(s-x_1)$). The set $K$ is a closed convex subset of $L^2(\Omega)$ whose boundary is given as follows:

$$\partial K = \left\{ q \in L^2(\Omega);\ \int_\Omega (BSB^*q)\cdot q\,dx = \eta \right\}.$$

Hence, for each $q \in \partial K$, the corresponding normal cone is the set

$$N_{\partial K}(q) = \{\rho BSB^*q;\ \rho \in \mathbb{R}\}.$$

In this context, we know that if $p \in L^2(\Omega)$ is a solution to (3), there exists a constant $\Lambda \in \mathbb{R}$ such that

$$\Lambda BSB^*p \in \partial g^*(p) - \partial(f^* \circ B^*)(p) = \partial g^*(p) - B\partial f^*(B^*p) \tag{4}$$

($\Lambda$ is a Lagrange multiplier associated to $p$). Let us set $u = SB^*p$ (so that $-\Delta u = B^*p$) and $\beta = \Lambda + 1$. Then, (4) can be written in the form

$$p \in H(\beta Bu - x_1). \tag{5}$$

If $\beta > 0$, taking $W = 1/\beta$, one finds that

$$p \in H(Bu - Wx_1). \tag{6}$$

Finally, since $p \in \partial K$, one also has

$$\int_\Omega (BSB^*q)\cdot q\,dx = \int_\Omega Bu\cdot p\,dx = \int_\Omega u\cdot B^*p\,dx = \int_\Omega |\nabla u|^2\,dx = \eta,$$

and consequently, $(u, W)$ is a solution of (2).

In the sequel, we make the assumption that

$$\eta < \int_\Omega |\nabla u_\alpha|^2\,dx \equiv F_\alpha,$$

where $u_\alpha$ is the unique solution to

$$\begin{aligned} -\Delta u &= \alpha \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Obviously, this is a necessary condition for the existence of a solution to (2).

[1] In [1, Section 6], one can find another choice of the functionals and spaces that leads to the same iterates.

### The Algorithms

In [1], the following algorithm is proposed to solve constrained problems of the kind (3):

(a) Fix $\lambda > 0$ and choose $p_0 \in H$.

(b) Then, for any given $k \geq 0$ and $p_k \in H$, compute $u_{k+1}$ and $p_{k+1}$ by solving

$$\begin{aligned} &u_{k+1} \in \partial f^* \left(B^* p_k\right) + B^{-1} N_{\partial K} \left(p_k\right), \\ &p_{k+1} = g'_\lambda \left(B u_{k+1} + \lambda p_k\right), \qquad p_{k+1} \in \partial K. \end{aligned} \tag{7}$$

Here, $g'_\lambda$ is the Yosida approximation to the maximal monotone operator $\partial g$, i.e.,

$$g'_\lambda = \frac{1}{\lambda} \left(Id - (Id + \lambda \partial g)^{-1}\right).$$

Let us briefly explain the meaning of (7). The component of $u_{k+1}$ in $B^{-1} N_{\partial K}(p_k)$ must be chosen so that $p_{k+1}$ verifies $p_{k+1} \in \partial K$.

In the context of (2), this algorithm reads as follows.

ALGORITHM 1.

(a) Fix $\lambda > 0$ and choose $p_0 \in L^2(\Omega)$. Compute $u_0 \in H_0^1(\Omega)$ by solving

$$\begin{aligned} -\Delta u &= p_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(b) Then, for any given $k \geq 0$, $p_k \in L^2(\Omega)$, and $u_k \in H_0^1(\Omega)$,

(b.1) compute $\beta_{k+1} \in \mathbb{R}$ by solving the equation

$$F_k(\beta) \equiv \int_\Omega \left|\nabla v_k(\beta)\right|^2 dx = \eta, \tag{8}$$

where $v_k(\beta)$ is (by definition) the solution to

$$\begin{aligned} -\Delta v &= H_\lambda \left(\beta u_k - x_1 + \lambda p_k\right) && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{9}$$

(b.2) Then, set

$$\begin{aligned} p_{k+1} &= H_\lambda \left(\beta_{k+1} u_k - x_1 + \lambda p_k\right), \\ u_{k+1} &= v_k \left(\beta_{k+1}\right). \end{aligned} \qquad \blacksquare$$

A variant of Algorithm 1 is given by the following iterates.

ALGORITHM 2.

(a) Fix $\overline{\lambda} > 0$ and choose $\overline{p}_0 \in L^2(\Omega)$. Compute $\overline{u}_0 \in H_0^1(\Omega)$ by solving

$$\begin{aligned} -\Delta \overline{u} &= \overline{p}_0 \quad \text{in } \Omega, \\ \overline{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(b) Then, for any given $k \geq 0$, $\overline{p}_k \in L^2(\Omega)$, and $\overline{u}_k \in H_0^1(\Omega)$,

(b.1) compute $W_{k+1} \in \mathbb{R}$ by solving the equation

$$\overline{F}_k(W) \equiv \int_\Omega \left|\nabla \overline{v}_k(W)\right|^2 dx = \eta, \tag{10}$$

where $\overline{v}_k(W)$ is the solution to

$$\begin{aligned} -\Delta \overline{v} &= H_{\overline{\lambda}} \left(\overline{u}_k - W x_1 + \overline{\lambda} \overline{p}_k\right) && \text{in } \Omega, \\ \overline{v} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{11}$$

(b.2) Then, set

$$\begin{aligned} \overline{p}_{k+1} &= H_{\overline{\lambda}} \left(\overline{u}_k - W_{k+1} x_1 + \overline{\lambda} \overline{p}_k\right), \\ \overline{u}_{k+1} &= \overline{v}_k \left(W_{k+1}\right). \end{aligned} \qquad \blacksquare$$

Let us prove that the iterates of Algorithm 2 are well defined; i.e., that for each $k$ there exists at least one solution $W_{k+1} \in \mathbf{R}$ to equation (10). The function $\overline{F}_k$ is continuous, nonincreasing, and bounded; indeed

$$0 \le \overline{F}_k(W) \le F_\alpha \equiv \int_\Omega |\nabla u_\alpha|^2 \, dx.$$

For $\epsilon > 0$, we denote by $\Omega_\epsilon$ the set

$$\Omega_\epsilon = \{x = (x_1, x_2)\,;\, x \in \Omega, \epsilon \le x_1\}.$$

We have

$$\overline{F}_k(W) = \int_\Omega H_{\overline{\lambda}}\left(\overline{u}_k + \overline{\lambda}\overline{p}_k - W x_1\right) \overline{v}_k(W)\, dx \le C(\Omega) \int_\Omega \left|H_{\overline{\lambda}}\left(\overline{u}_k + \overline{\lambda}\overline{p}_k - W x_1\right)\right|^2 dx,$$

where $C(\Omega)$ is a constant depending only on $\Omega$. For any given $\rho > 0$, let $\epsilon > 0$ be such that

$$\operatorname{meas}(\Omega - \Omega_\epsilon) < \frac{\rho}{C(\Omega)\alpha^2}.$$

Then, if $W \ge W^*(\epsilon) = 1/\epsilon\,(\max_{x\in\Omega} u_\alpha(x) + \overline{\lambda}\alpha)$, one has

$$H_{\overline{\lambda}}\left(\overline{u}_k + \overline{\lambda}\overline{p}_k - W x_1\right) = 0 \quad \text{in } \Omega_\epsilon,$$

and consequently,

$$\overline{F}_k(W) \le C(\Omega) \int_{\Omega - \Omega_\epsilon} \left|H_{\overline{\lambda}}\left(u_\alpha + \overline{\lambda}\alpha - W x_1\right)\right|^2 dx \le C(\Omega)\,\alpha^2 \operatorname{meas}(\Omega - \Omega_\epsilon) < \rho,$$

i.e., $\lim_{W\to+\infty} \overline{F}_k(W) = 0$.

On the other hand, for $W \le W_*(\epsilon) = (-\lambda\alpha)/\epsilon$, one has

$$H_{\overline{\lambda}}\left(\overline{u}_k + \overline{\lambda}\overline{p}_k - W x_1\right) = \alpha \quad \text{in } \Omega_\epsilon.$$

Let us denote by $z_{\alpha,\epsilon}$, the solution to

$$\begin{aligned} -\Delta z &= \chi_{\alpha,\epsilon} && \text{in } \Omega, \\ z &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\chi_{\alpha,\epsilon}$ is $\alpha$ times the characteristic function of $\Omega_\epsilon$. Then, for all $W \le W_*(\epsilon)$, one has $\overline{v}_k(W) \ge z_{\alpha,\epsilon}$ in $\Omega$, and

$$\overline{F}_k(W) \ge \int_{\Omega_\epsilon} \alpha z_{\alpha,\epsilon}\, dx = F_\alpha - \left( \int_\Omega \alpha u_\alpha \, dx - \int_{\Omega_\epsilon} \alpha z_{\alpha,\epsilon}\, dx \right).$$

Now, for given $\rho > 0$, we can choose $\epsilon > 0$, so that $\int_\Omega \alpha u_\alpha\, dx - \int_{\Omega_\epsilon} \alpha z_{\alpha,\epsilon}\, dx < \rho$. Therefore,

$$\lim_{W\to-\infty} \overline{F}_k(W) = F_\alpha.$$

It is now clear, that if $0 < \eta < F_\alpha$, there exists at least one solution $W_{k+1} \in \mathbf{R}$ to equation (10) (this argument also shows that $\{W_{k+1}\}$ is uniformly bounded).

Let us mention that concerning Algorithm 1, it can be proved that for each $k$ there exists $\lambda$ such that (8) possesses a solution.

**Finite Element Approximation**

The main difficulty in the algorithms above is the computation of the solutions to equations (8) and (10). Indeed, this requires the numerical solution of possibly many Dirichlet problems of kinds (9) or (11). This can be made easier by using finite element techniques. We shall use a $P_1$-Lagrange (piecewise linear) finite element approximation. Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ and let $\{a_i\}_{i=1}^n$ be the set of the corresponding nodal points. For the sake of simplicity in the exposition, we shall suppose that they have been numbered in such a way that the first $n_0$ points belong to $\Omega$ (this is not essential for the following).

Consider the finite dimensional space

$$V_h = \left\{v_h;\ v_h \in C^0\left(\overline{\Omega}\right),\ v_h\,|_T \in P_1,\ \forall T \in \mathcal{T}_h\right\},$$

and its subspace

$$V_h^0 = \{v_h;\ v_h \in V_h,\ v_h = 0 \text{ on } \partial\Omega\}.$$

Of course, $V_h$ and $V_h^0$ must be viewed as approximations to $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively. Then, it is well known that a function $v_h \in V_h$, (respectively, $w_h \in V_h^0$) is uniquely determined by the values $v_h(a_k)$, for $k = 1, 2, \ldots, n$. (Respectively, the values $w_h(a_k)$, for $k = 1, 2, \ldots, n_0$.) It is thus customary to introduce the canonical basis $\{\varphi_1, \ldots, \varphi_n\}$ of $V_h$, where

$$\varphi_i \in V_h \quad \text{and} \quad \varphi_i\left(a_j\right) = \delta_{ij}, \qquad \forall i, j = 1, 2, \ldots, n.$$

Accordingly, the canonical basis of $V_h^0$ is $\{\varphi_1, \ldots, \varphi_{n_0}\}$. Observe that

$$v_h = \sum_{i=1}^{n} v_h\left(a_i\right)\varphi_i, \qquad \forall v_h \in V_h \quad \text{and} \quad w_h = \sum_{i=1}^{n_0} w_h\left(a_i\right)\varphi_i, \qquad \forall w_h \in V_h^0.$$

For practical purposes, we shall identify any function $v_h \in V_h$, (respectively, $w_h \in V_h^0$) with the corresponding vector $\overline{v} \in \mathbb{R}^n$, $\overline{v} = (v_i)_{i=1}^n$, $v_i = v_h(a_i)$, (respectively, $\overline{w} \in \mathbb{R}^{n_0}$, $\overline{w} = (w_i)_{i=1}^{n_0}$, $w_i = w_h(a_i)$).

Let us now explain how a numerical approximation to the solution to the scalar equations (8) and (10) can be found. To fix ideas, we will only speak of equation (10). Let us denote by $\overline{P}_k(W)$, the second member in the Dirichlet problem (11), i.e., $\overline{P}_k(W) = H_{\overline{\lambda}}(\overline{u}_k - Wx_1 + \overline{\lambda}\overline{p}_k)$. The scalar equation (10) now converts into

$$\overline{F}_{h,k}(W) \equiv \int_{\Omega} \overline{P}_{h,k}(W)\overline{v}_{h,k}(W)\,dx = \eta, \tag{12}$$

where $\overline{v}_{h,k}(W)$ verifies

$$\begin{aligned} &\overline{v}_{h,k}(W) \in V_h^0, \\ &\int_{\Omega} \nabla\overline{v}_{h,k}(W)\nabla\varphi_i\,dx, = \int_{\Omega} \overline{P}_{h,k}(W)\varphi_i\,dx, \qquad \forall i = 1, \ldots, n_0, \end{aligned} \tag{13}$$

and $\overline{P}_{h,k}(W)$ is

$$\overline{P}_{h,k}(W) \in V_h, \qquad \overline{P}_{h,k}(W)\left(a_i\right) = \overline{H}_{\overline{\lambda}}\left(\overline{u}_i^k - Wx_{i,1} + \overline{\lambda}\overline{p}_i^k\right), \qquad \forall i = 1, 2, \ldots n \tag{14}$$

(here, $(x_{i,1}, x_{i,2})$ stands for the coordinates of the $i^{\text{th}}$ nodal point).

The function $\overline{F}_{h,k}(W)$ is nonincreasing, so that equation (12) is easily solvable by an iterative method. Of course, this needs, at each step, the computation of the solution to (13) for a given value of $W$. It is readily seen that this is just to solve the $n_0$-dimensional linear system $A\overline{v}_k(W) = b_k(W)$, where $A$ and $b_k(W)$ are given as follows:

$$A = (a_{ij})_{i,j=1}^{n_0}, \qquad a_{ij} = \int_\Omega \nabla\varphi_i \nabla\varphi_j \, dx,$$
$$b_k(W) = \left(b_i^k(W)\right)_{i=1}^{n_0}, \qquad b_i^k(W) = \int_\Omega \overline{P}_{h,k}(W)\varphi_i \, dx.$$

The matrix $A$, which is common to all the iterates, is symmetric and definite positive. It is also a sparse matrix, and if an appropriate renumerotation of the nodal points is performed, all nonvanishing components will be near the diagonal line. Consequently, it will be easy to solve all the previous linear systems via Cholesky's method. Of course, the triangular matrix $L$ which provides the factorization $A = LL^t$ must be computed at the beginning of the program.

In short, at each iteration of the algorithm for solving the scalar equation (12), for a given value of $W$, the computations to carry out are:

(a) Compute $\overline{P}_{h,k}(W)$ given by (14).
(b) Compute the second member $b_k(W)$.
(c) Solve the linear triangular systems $Ly = b_k(W)$ and $L^t\overline{v}_k(W) = y$.
(d) Compute the value of $\overline{F}_{h,k}(W)$, i.e., the integral $\int_\Omega \overline{P}_{h,k}(W)\overline{v}_{h,k}(W)\, dx$.

### Numerical Experiences with Algorithm 2

For the numerical tests, we have taken $\Omega = (0,10) \times (-10,10)$ and $\eta = 3$. We have used a mesh with 1375 points and 2608 triangles, which are smaller in the region occupied by the vortex (see Figure 1). We have made several computations which correspond to different values of the parameter $\alpha$.

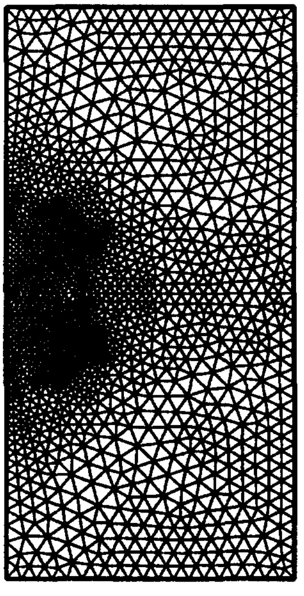

Figure 1. Triangulation of the domain $\Omega$. Number of triangles: 2608. Number of nodal points: 1375.

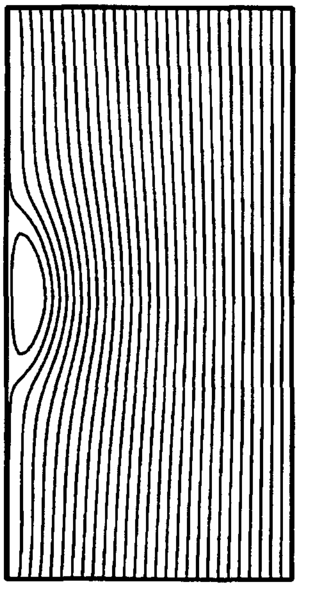

Figure 2. Isolines of the stream function $\psi = u - Wx_1$, for the data $\alpha = 1$, $\eta = 3$.

We have initialized the iterates with the constant function $p_0(x) \equiv 1$ in all cases. The convergence criterium has been

$$\|p_{k+1} - p_k\| < \varepsilon = 10^{-5}.$$

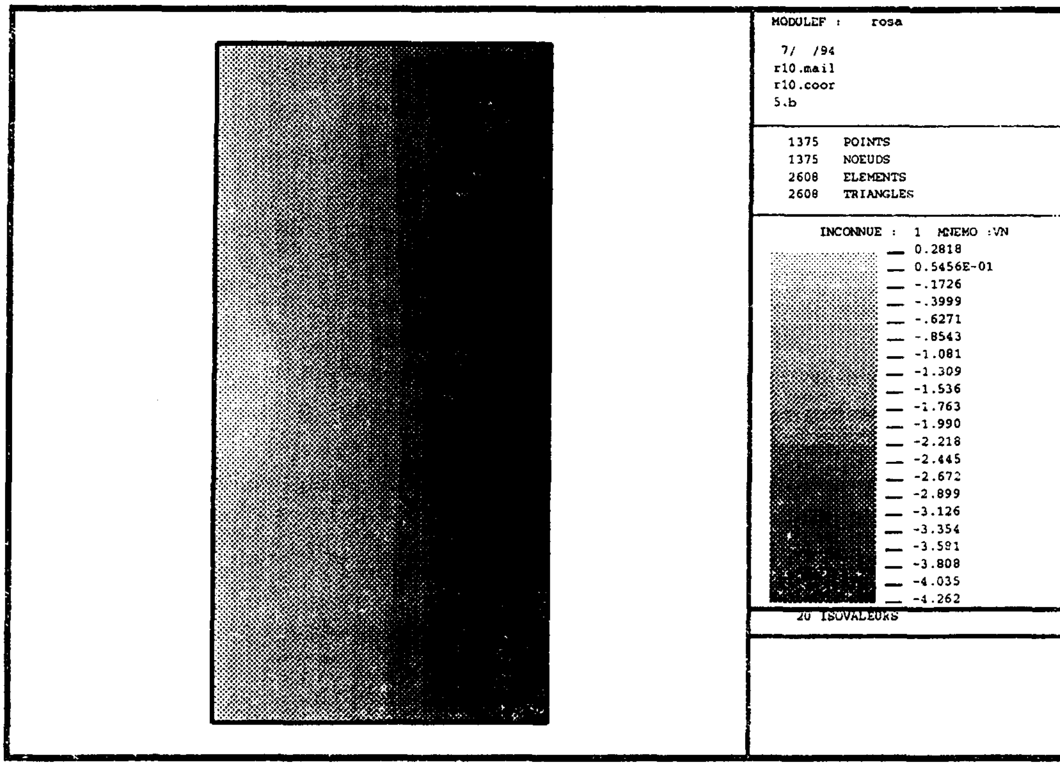


Figure 3. Representation of the values of the computed function $\psi = u - Wx_1$, for the data $\alpha = 1$, $\eta = 3$.

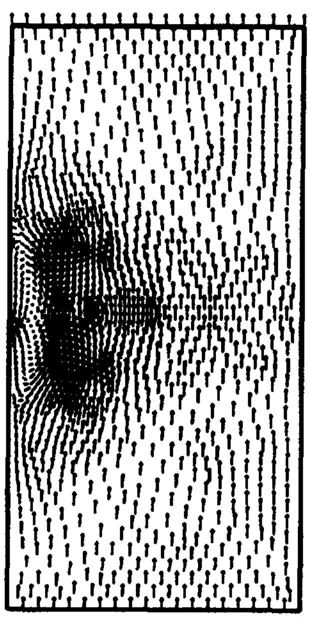

Figure 4. Representation of the velocity field $\vec{v} = \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}\right)$.

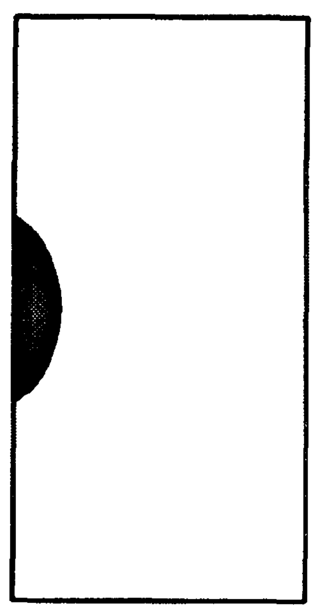

Figure 5. Region occupied by the vortex, i.e., $A = \{x \in \Omega;\ \psi(x) > 0\}$, for the data $\alpha = 1$, $\eta = 3$.

We present in Figures 2–5, different views of the stream function $\psi = u - Wx_1$ in the case $\alpha = 1$, obtained with $\lambda = 10^{-6}$. Figures 6–11 show the results in the cases $\alpha = 1.5$, $\alpha = 2$, and $\alpha = 4$, obtained with $\lambda = 10^{-7}$.

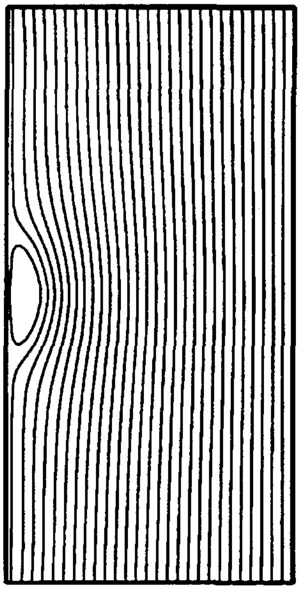

Figure 6. Stream lines for the data $\alpha = 1.5$, $\eta = 3$. Computed value of $W = 0.5303524$.

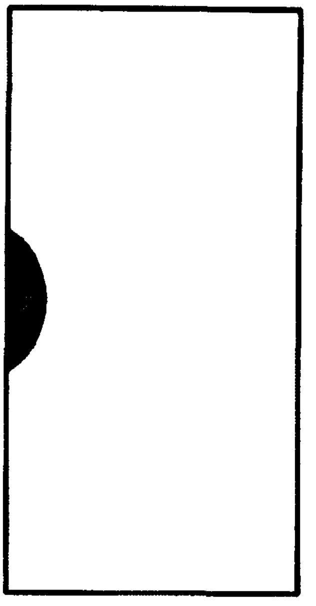

Figure 7. Vorticity region for the same case.

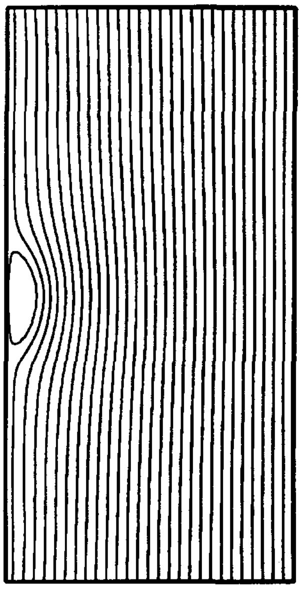

Figure 8. Stream lines for the data $\alpha = 2$, $\eta = 3$. Computed value of $W = 0.5868746$.

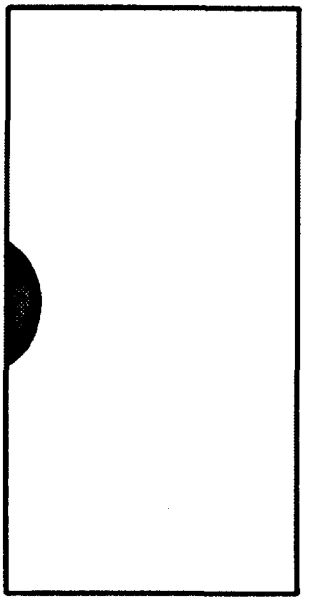

Figure 9. Vorticity region for the same case.

## Comparison of the Algorithms

In accordance with our tests, the behaviour of Algorithm 2 is much more stable. It always converges if the value of $\lambda$ is small enough ($\lambda \leq 0.001$). Furthermore, the number of iterates needed for convergence in the case of Algorithm 1 is too sensitive to the initial value $W$. Furthermore, Algorithm 1 also needs for convergence to be more accurate in the solution of scalar equations. This is not surprising, because the right-hand sides in the Dirichlet problems in Algorithm 1 are much more nonlinear than those in Algorithm 2. In the cases where both algorithms converge, the results are similar.

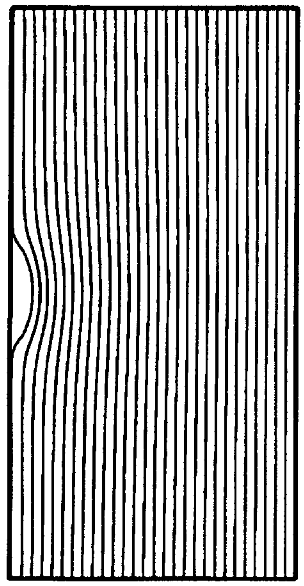

Figure 10. Stream lines for the data $\alpha = 4$, $\eta = 3$. Computed value of $W =$ 0.8348695.

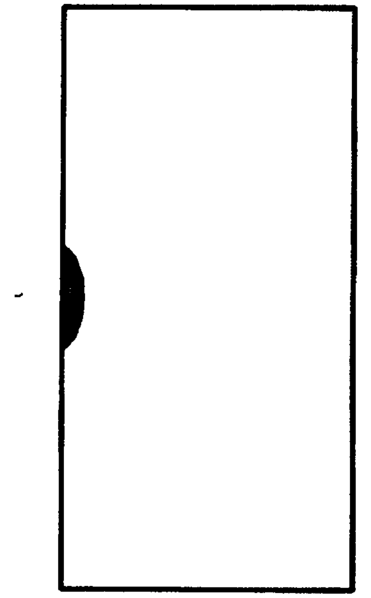

Figure 11. Vorticity region for the same case.

Table 1. Dependence of the algorithms on the value of $\lambda$, for an initialization $W_0 =$ 0.3.

| | Algorithm 1 | | Algorithm 2 | |
|---|---|---|---|---|
| $\lambda$ | N. Iter. | $W_0$ | N. Iter. | $W_0$ |
| $10^{-3}$ | — | — | — | — |
| $10^{-4}$ | — | — | 13 | 0.43225139 |
| $10^{-5}$ | 9 | 0.42134008 | 9 | 0.42134380 |
| $10^{-6}$ | — | — | 9 | 0.42623562 |
| $10^{-7}$ | — | — | 9 | 0.42623562 |

Table 2. Dependence of algorithms on the initialization of $W_0$ for $\lambda = 10^{-7}$.

| | Algorithm 1 | | Algorithm 2 | |
|---|---|---|---|---|
| $W_0$ | N. Iter. | $W$ | N. Iter. | $W$ |
| 0.1 | 9 | 0.42134279 | 9 | 0.42138773 |
| 0.2 | 9 | 0.42134270 | 9 | 0.42139536 |
| 0.25 | — | — | 9 | 0.42136800 |
| 0.3 | — | — | 9 | 0.42623562 |
| 0.32 | 9 | 0.42134058 | 9 | 0.42134523 |
| 0.4 | 9 | 0.42134362 | 8 | 0.42134398 |
| 0.5 | — | — | 8 | 0.42135161 |
| 0.57 | 9 | 0.42134041 | 9 | 0.42134762 |
| 0.6 | — | — | 15 | 0.41447192 |
| 0.62 | 9 | 0.42134339 | 9 | 0.42627427 |
| 0.7 | 9 | 0.42134804 | 9 | 0.42134368 |

## 4. THE NONVARIATIONAL CASE: FREE FLUX PARAMETER

In this section, the problem under consideration is the following:

$$\text{Find } u \in H_0^1(\Omega) \quad \text{and} \quad Z \in \mathbb{R} \text{ such that} \tag{15}$$

$$-\Delta u(x) \in H\left(u(x) - Wx_1 - Z\right) \quad \text{a.e. in } \Omega, \qquad x = (x_1, x_2), \qquad \text{(cont. 15)}$$
$$\int_\Omega |\nabla u|^2\, dx = \eta.$$

As we have already said in Section 2, we cannot find a variational formulation for (15). Nevertheless, based on the iterates proposed and tested for problem (1) in Section 3, we propose here the following, analogous to Algorithm 2, for problem (15).

ALGORITHM 3.

(a) Fix $\lambda > 0$ and choose $p_0 \in L^2(\Omega)$. Compute $u_0 \in H_0^1(\Omega)$ by solving

$$\begin{aligned} -\Delta u &= p_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(b) Then, for any given $k \geq 0$, $p_k \in L^2(\Omega)$, and $u_k \in H_0^1(\Omega)$,

(b.1) compute $Z_{k+1} \in \mathbf{R}$ by solving the equation

$$G_k(Z) \equiv \int_\Omega |\nabla \widehat{v}_k(Z)|^2\, dx = \eta, \tag{16}$$

where $\widehat{v}_k(Z)$ is the solution of

$$\begin{aligned} -\Delta \widehat{v} &= H_\lambda\left(u_k - Wx_1 - Z + \lambda p_k\right) && \text{in } \Omega, \\ \widehat{v} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{17}$$

(b.2) Then, set

$$\begin{aligned} p_{k+1} &= H_\lambda\left(u_k - Wx_1 - Z_{k+1} + \lambda p_k\right), \\ u_{k+1} &= \widehat{v}_k\left(Z_{k+1}\right). \end{aligned}$$ ∎

To prove that this algorithm is well defined, we will show that, for fixed $W$ and with $u_k$ and $p_k$ being given by the algorithm, there exists at least one solution to equation (16). The function $G_k(Z)$ in (16) can also be written in the form

$$G_k(Z) \equiv \int_\Omega H_\lambda\left(u_k - Wx_1 - Z + \lambda p_k\right) \widehat{v}_k(Z)\, dx.$$

Then, it is clear that $G_k(Z)$ is continuous, nonincreasing, and bounded, because

$$0 \leq G_k(Z) \leq F_\alpha, \qquad \forall\, k \geq 0, \quad \forall\, Z \in \mathbb{R}.$$

We also have the inequalities

$$-Wa \leq u_k - Wx_1 + \lambda p_k \leq \max_{x\in\Omega} u_\alpha + \lambda\alpha.$$

Consequently, there exist $Z^* = \max_{x\in\Omega} u_\alpha + \lambda\alpha$ and $Z_* = -Wa - \lambda\alpha$ such that, $G_k(Z) = 0$ for $Z \geq Z^*$ and $G_k(Z) = F_\alpha$ for $Z \leq Z_*$.

Therefore, since $0 < \eta < F_\alpha$, equation (16) always possesses at least one solution. Once again, we see that the sequence $\{Z_k\}$ is uniformly bounded.

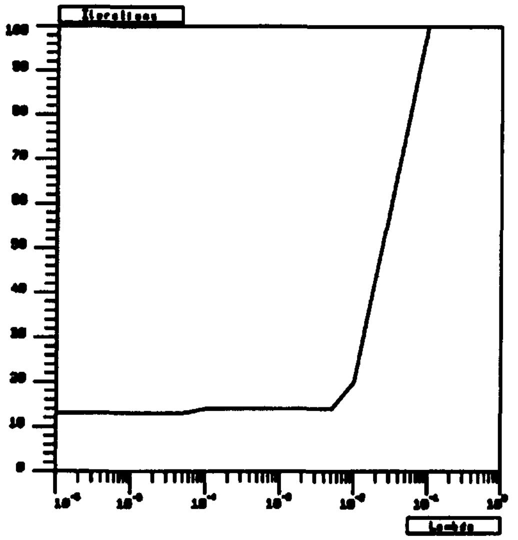

Figure 12. Dependence of the number of iterations on $\lambda$.

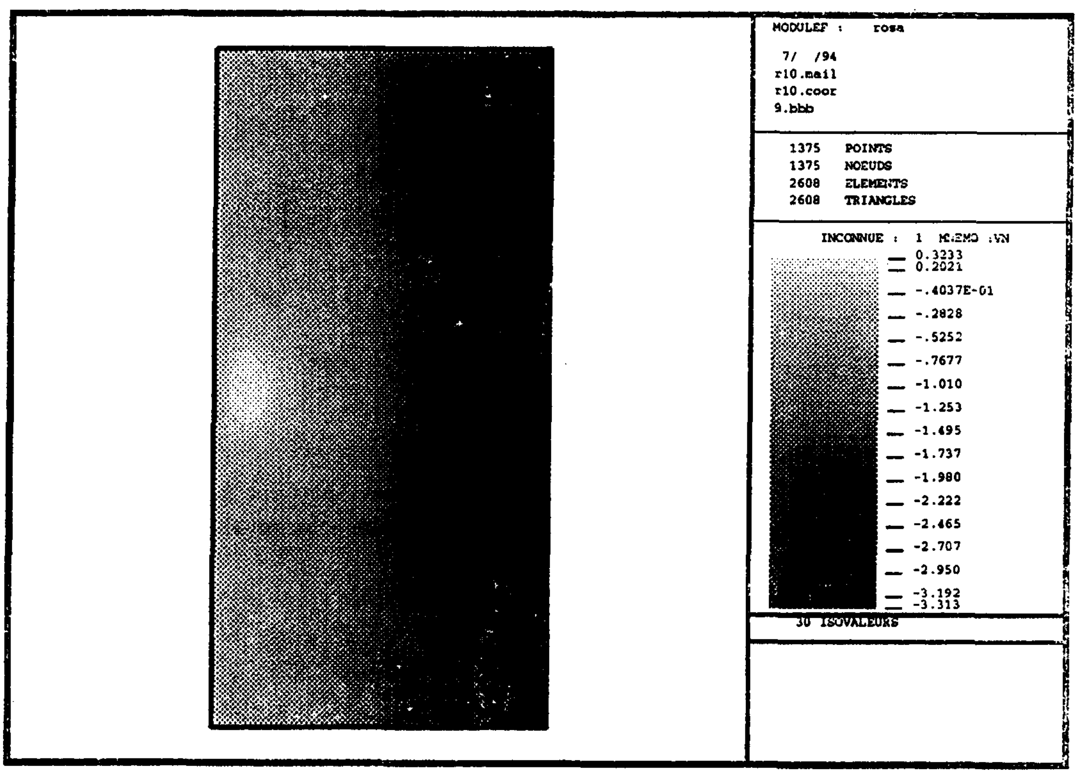


Figure 13. Representation of the values of the function $\psi = u - Wx_1 - Z$, for the data $\alpha = 1$, $\eta = 3$, $W = 0.32$. Computed value of $Z = 0.11324509$.

### 4.1. Numerical Experiences

For the numerical experiences, we have used the same triangulation as that in Section 3 (see Figure 1). We visualize here the results for the data $\alpha = 1$, $\eta = 3$, $W = 0.32$. In Figure 12, a relationship between the value of $\lambda$ and the number of iterations needed is shown. Figures 13–15 are concerned with the stream function $\psi = u - Wx_1 - Z$ and, in Figure 16, the velocity field $\vec{v} = (\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1})$ is displayed. Finally, in Figure 17, we show the graph of the computed function $Z = Z(W)$.

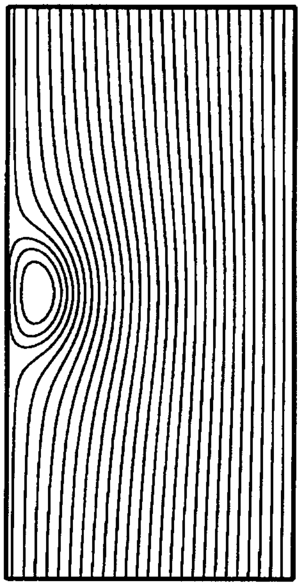

Figure 14. Streamlines for the same data.

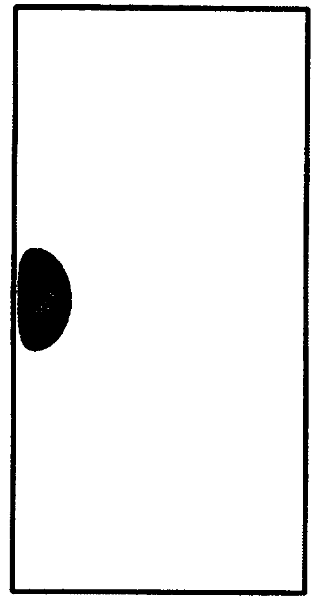

Figure 15. Vorticity region for the same case, i.e., $A = \{x \in \Omega;\ \psi = u - Wx_1 - Z > 0\}$.

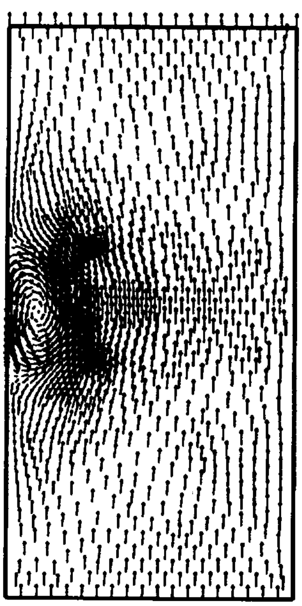

Figure 16. Representation of the velocity field, $\vec{v} = (\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1})$.

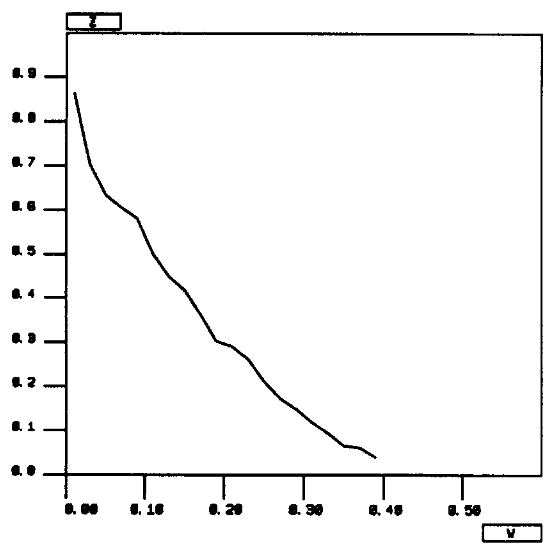


Figure 17. Graph of the computed function $Z = Z(W)$.

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